

Symmetry-breaking bifurcation for the one-dimensional Liouville type equation

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Abstract. The two-point boundary value problem for the one-dimensional Liouville type equation

$$\begin{cases} u'' + \lambda|x|^l e^u = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0 \end{cases}$$

is considered, where $\lambda > 0$ and $l > 0$. In this paper, a symmetry-breaking result is obtained by using the Morse index. The problem

$$\begin{cases} u'' + \lambda|x|^l (u+1)^p = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0 \end{cases}$$

is also considered, where $\lambda > 0$, $l > 0$, $p > 1$ and $(p-1)l > 4$.

Key words and phrases: symmetry-breaking bifurcation, positive solution, one-dimensional Liouville type equation, Morse index, Korman solution

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1. INTRODUCTION

In this paper we consider the two-point boundary value problem for the one-dimensional Liouville type equation

$$(1.1) \quad \begin{cases} u'' + \lambda|x|^l e^u = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ and $l > 0$.

Jacobsen and Schmitt [7] presented the exact multiplicity result of radial solutions for the multi-dimensional problem

$$(1.2) \quad \begin{cases} \Delta u + \lambda|x|^l e^u = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

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where $\lambda > 0$, $l \geq 0$, $B := \{x \in \mathbf{R}^N : |x| < 1\}$ and $N \geq 1$. In the case $N = 1$, problem (1.2) is reduced to (1.1). We note here that every solution of (1.2) is positive in B , by the strong maximum principle. Jacobsen and Schmitt [7] proved the following (i)–(iii):

- (i) if $1 \leq N \leq 2$, then there exists $\lambda_* > 0$ such that (1.2) has exactly two radial solutions for $0 < \lambda < \lambda_*$, a unique radial solution for $\lambda = \lambda_*$ and no radial solution for $\lambda > \lambda_*$;
- (ii) if $3 \leq N < 10 + 4l$, then (1.2) has infinitely many radial solutions when $\lambda = (l+2)(N-2)$ and a finite but large number of radial solutions when $|\lambda - (l+2)(N-2)|$ is sufficiently small;
- (iii) if $N \geq 10 + 4l$, then (1.2) has a unique radial solution for $0 < \lambda < (l+2)(N-2)$ and no radial solution for $\lambda \geq (l+2)(N-2)$.

Recently, Korman [14] gave an alternative proof of (i)–(iii), and his method is very interesting and easy to understand it. Results (i)–(iii) were established by Liouville [16], Gel’fand [4], Joseph and Lundgren [8] for problem (1.2) with $l = 0$, that is,

$$(1.3) \quad \begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\Omega = B$.

A celebrated theorem by Gidas, Ni and Nirenberg [5] shows that every positive solution of (1.3) is radially symmetric when $\Omega = B$. However, when Ω is an annulus $A := \{x \in \mathbf{R}^N : a < |x| < b\}$, $a > 0$, problem (1.3) may have non-radial solutions. Indeed, Nagasaki and Suzuki [18] found that large non-radial solutions of (1.3) when $N = 2$ and $\Omega = A$. More precisely, for each sufficiently large $\mu > 0$, there exist $\lambda > 0$ and a non-radial solution u of (1.3) such that $\int_A e^u dx = \mu$ when $N = 2$ and $\Omega = A$. Lin [15] showed that (1.3) has infinitely many symmetry-breaking bifurcation points when $N = 2$ and $\Omega = A$. Dancer [3] proved that non-radial solution branches emanating from the symmetry-breaking bifurcation points found by Lin [15] are unbounded. Kan [9, 10] considered (1.3) with $\Omega = A$ and $N = 2$ and investigated the structure of non-radial solutions bifurcating from radial solutions in the case where a is sufficiently small. More general potential and domain were considered by del Pino, Kowalczyk and Musso [2], and they constructed concentrating solutions.

Recently, Miyamoto [17] proved the following result for (1.2).

Theorem A ([17]). *Let n_0 be the largest integer that is smaller than $1 + \frac{l}{2}$ and let $\alpha_n := 2 \log \frac{2l+4}{l+2-2n}$. All the radial solutions of (1.2) with*

$N = 2$ can be written explicitly as

$$\lambda(\alpha) = 2(l+2)^2(e^{-\alpha/2} - e^{-\alpha}), \quad U(r; \alpha) = \alpha - 2\log(1 + (e^{\alpha/2} - 1)r^{l+2}).$$

The radial solutions can be parameterized by the L^∞ -norm, it has one turning point at $\lambda = \lambda(\alpha_0) = (l+2)/2$, and it blows up as $\lambda \downarrow 0$. For each $n \in \{1, 2, \dots, n_0\}$, $(\lambda(\alpha_n), U(r; \alpha_n))$ is a symmetry breaking bifurcation point from which an unbounded branch consisting of non-radial solutions of (1.2) with $N = 2$ emanates, and $U(r; \alpha)$ is non-degenerate if $\alpha \neq \alpha_n$, $n = 0, 1, \dots, n_0$. Each non-radial branch is in $(0, \lambda(\alpha_0)) \times \{u > 0\} \subset \mathbf{R} \times H_0^2(B)$.

We return to problem (1.1). Korman [14] found the interesting property of radial solutions to (1.2). We will use it for the case $N = 1$. Let w be a unique solution of the initial value problem

$$\begin{cases} w'' + |x|^l e^w = 0, & x > 0, \\ w(0) = w'(0) = 0. \end{cases}$$

It is easy to show that

$$(1.4) \quad w(x) < 0, \quad w'(x) < 0, \quad w''(x) < 0 \quad \text{for } x > 0$$

and $\lim_{x \rightarrow \infty} w(x) = -\infty$. Hence, there exists the inverse function η of $-w(x)$. It follows that $\eta \in C^2(0, \infty)$, $\eta(t) > 0$, $\eta'(t) > 0$ for $t > 0$, $\eta(0) = 0$, and $\lim_{t \rightarrow \infty} \eta(t) = \infty$. We set

$$(1.5) \quad \lambda(\alpha) = [\eta(\alpha)]^{l+2} e^{-\alpha}$$

and

$$(1.6) \quad U(x; \alpha) = w(\eta(\alpha)|x|) + \alpha.$$

By a direct calculation, we easily prove that, for each $\alpha > 0$, $U(x; \alpha)$ satisfies $\|U\|_\infty = \alpha$ and is a positive even solution of (1.1) at $\lambda = \lambda(\alpha)$. Here and hereafter we use the notation: $\|u\|_\infty = \sup_{x \in [-1, 1]} u(x)$.

Lemma 1.1 (Korman [14]). *For each $\alpha > 0$, $U(x; \alpha)$ is a positive even solution of (1.1) at $\lambda = \lambda(\alpha)$ and $\|U\|_\infty = \alpha$.*

The author would like to call $U(x; \alpha)$ the *Korman solution* of (1.1). Korman [14] also presented this kind of radial solutions to

$$\begin{cases} \Delta u + \lambda|x|^l f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

in the following cases: $f(u) = (u+1)^p$, $p > 1$; $f(u) = (1-u)^{-p}$, $p > 1$; $f(u) = e^{-u}$. By using Lemma 1.1, we can show the following result, which will be shown in Section 2.

Proposition 1.1. *The functions $\lambda(\alpha)$ and $U(x; \alpha)$ satisfy $\lambda(\alpha) \in C^2(0, \infty)$, $U(x; \alpha) \in C^2([-1, 1] \times (0, \infty))$, and*

$$(1.7) \quad \lim_{\alpha \rightarrow +0} \lambda(\alpha) = \lim_{\alpha \rightarrow \infty} \lambda(\alpha) = 0.$$

Moreover, there exists $\alpha_ > 0$ such that $\lambda'(\alpha) > 0$ for $0 < \alpha < \alpha_*$, $\lambda(\alpha_*) = 0$ and $\lambda'(\alpha) < 0$ for $\alpha > \alpha_*$.*

Hereafter, let α_* be as in Proposition 1.1.

Let $m(\alpha)$ be the Morse index of $U(x; \alpha)$, that is, the number of negative eigenvalues μ to

$$(1.8) \quad \begin{cases} \phi'' + \lambda(\alpha)|x|^l e^{U(x; \alpha)} \phi + \mu \phi = 0, & x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

A solution $U(x; \alpha)$ is said to be degenerate if $\mu = 0$ is an eigenvalue of (1.8). Otherwise, it is said to be nondegenerate.

We denote by $\mu_k(\alpha)$ the k -th eigenvalue of (1.8). We recall that

$$\mu_1(\alpha) < \mu_2(\alpha) < \cdots < \mu_k(\alpha) < \mu_{k+1}(\alpha) < \cdots, \quad \lim_{k \rightarrow \infty} \mu_k(\alpha) = \infty,$$

no other eigenvalues, an eigenfunction ϕ_k corresponding to $\mu_k(\alpha)$ is unique up to a constant, and ϕ_k has exactly $k - 1$ zeros in $(-1, 1)$. We find that $\mu_k \in C(0, \infty)$. (See, for example, [11].)

The following theorem is the main result of this paper.

Theorem 1.1. *Let $(\lambda(\alpha), U(x; \alpha))$ be as in (1.5)–(1.6) and let $\alpha_* > 0$ be as in Proposition 1.1. Then there exist constants α_1, α_2 and α_3 such that $\alpha_* < \alpha_1 \leq \alpha_2 \leq \alpha_3$ and the following (i)–(vii) hold:*

- (i) *if $0 < \alpha < \alpha_*$, then $m(\alpha) = 0$ and $U(x; \alpha)$ is nondegenerate;*
- (ii) *if $\alpha = \alpha_*$, then $m(\alpha) = 0$ and $U(x; \alpha)$ is degenerate;*
- (iii) *if $\alpha_* < \alpha < \alpha_1$, then $m(\alpha) = 1$ and $U(x; \alpha)$ is nondegenerate;*
- (iv) *if $\alpha = \alpha_1$, then $m(\alpha) = 1$ and $U(x; \alpha)$ is degenerate;*
- (v) *if $\alpha = \alpha_2$, then $m(\alpha) = 1$, $U(x; \alpha)$ is degenerate and the point $(\lambda(\alpha_2), U(x; \alpha_2))$ is a non-even bifurcation point, that is, for each $\varepsilon > 0$, there exists (λ, u) such that u is a positive non-even solution of (1.1) and $|\lambda - \lambda(\alpha_2)| + \|u - U(\cdot, \alpha_2)\|_\infty < \varepsilon$;*
- (vi) *if $\alpha = \alpha_3$, then $m(\alpha) = 1$ and $U(x; \alpha)$ is degenerate;*
- (vii) *if $\alpha > \alpha_3$, then $m(\alpha) = 2$ and $U(x; \alpha)$ is nondegenerate.*

Moreover, if $0 < \lambda < \lambda(\alpha_3)$, then (1.1) has a positive non-even solution u which satisfies $\lim_{\lambda \rightarrow +0} \|u\|_\infty = \infty$.

We note here that if u is a non-even solution of (1.1), then so is $u(-x)$.

It is natural to expect that the following conjecture is true.

Conjecture 1.1. *In Theorem (1.1), $\alpha_1 = \alpha_2 = \alpha_3$.*

Recalling the result by Jacobsen and Schmitt [7], the structures of radial solutions of (1.2) with $N = 2$ and even solutions of (1.1) seem to be same. However, in [17] Miyamoto proved that the Morse index of the radial solution increases by one when α passes each α_n , $n = 0, 1, 2, \dots, n_0$, where α_n is as in Theorem A. On the other hand, by Lemma 2.3 below, the Morse index of even solutions of (1.2) is at most 2 for each $l > 0$.

When $N = 2$, radial solutions of problems (1.2) and (1.3) can be written explicitly, and hence, Lin [15] and Miyamoto [17] succeeded to find the bifurcation points. That is difficult even if we know exact solutions, much more difficult if we do not know them. When $N \neq 2$, we do not know exact radial solutions of (1.2) with $l > 0$. However, recently Korman [14] found the solution (1.6). When $N = 1$, the structure of eigenvalues $\{\mu_k(\alpha)\}_{k=1}^\infty$ of (1.8) is well-known. Combining these facts, we can show (i)–(iii) of Theorem 1.1.

Now we set

$$\psi(x; \alpha) := xU'(x; \alpha) + l + 2 = \eta(\alpha)|x|w'(\eta(\alpha)|x|) + l + 2$$

It is easy to check that the following result holds.

Lemma 1.2. *The function $\psi(x; \alpha)$ is a solution of the linearized equation*

$$(1.9) \quad \psi'' + \lambda(\alpha)|x|^l e^{U(x; \alpha)} \psi = 0.$$

Lemma 1.2 was found by Korman [12] when $l = 0$. See also [13, Proposition 2.2] and [14, Lemma 5.1]. From Lemma 1.2, it follows that $m(\alpha) \leq 2$ for $\alpha > 0$. See Lemma 2.3 below. Moreover, by using the comparison function

$$y(x) = xU(x; \alpha) - (x - 1)^2 U'(x; \alpha),$$

which was introduced in [19], we can prove that $m(\alpha) \geq 2$ for all sufficiently large $\alpha > 0$. See Lemma 3.1 below. Then we can find a symmetry-breaking bifurcation point of (1.1), by using the Leray-Schauder degree, and hence we will obtain (iv)–(vii) of Theorem 1.1.

By using similar argument, we can establish a symmetry-breaking bifurcation result for the problem

$$(1.10) \quad \begin{cases} u'' + \lambda|x|^l(u+1)^p = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$, $l > 0$ and $p > 1$.

Proposition 1.2. *There exists $\lambda_* > 0$ such that (1.10) has exactly two positive even solutions for $0 < \lambda < \lambda_*$, a unique positive even solution for $\lambda = \lambda_*$, and no positive even solution for $\lambda > \lambda_*$.*

Proposition 1.3. *Proposition 1.1 remains valid if (1.1) is replaced by (1.10).*

Theorem 1.2. *Assume that $(p-1)l > 4$. Theorem 1.1 remains valid if (1.1) is replaced by (1.10).*

The proofs of Propositions 1.2, 1.3 and Theorem 1.2 will be given in Section 6. In Section 2 we prove Proposition 1.1 and study the eigenvalues $\mu_1(\alpha)$ and $\mu_3(\alpha)$. In Section 3 we study $\mu_2(\alpha)$. In Section 4 we give a criterion for the existence of one more positive solution. In Sections 5, we give a proof of Theorem 1.1.

2. THE FIRST AND THIRD EIGENVALUES

In this section we study eigenvalues $\mu_1(\alpha)$ and $\mu_3(\alpha)$ of the linearized problem (1.8). We recall Lemmas 1.1 and 1.2. First we show Proposition 1.1.

Proof of Proposition 1.1. We recall (1.4). Since $w(0) = w'(0) = w''(0) = 0$ and $\eta \in C^2(0, \infty)$, we conclude that $\lambda(\alpha) \in C^2(0, \infty)$ and $U(x; \alpha) \in C^2([-1, 1] \times (0, \infty))$. It is easy to see that $\lim_{\alpha \rightarrow +0} \lambda(\alpha) = 0$. Since $w''(x) < 0$ for $x > 0$, we have

$$w'(x) \leq w'(1) < 0, \quad x \geq 1.$$

Integrating this inequality on $[1, x]$, we obtain

$$w(x) \leq w(1) + w'(1)(x-1) \leq -c(x-1), \quad x \geq 1,$$

where $c = -w'(1) > 0$. Letting $x = \eta(\alpha)$, we find that

$$0 < \lambda(\alpha) = [\eta(\alpha)]^{l+2} e^{-\alpha} = x^{l+2} e^{w(x)} \leq x^{l+2} e^{-c(x-1)}, \quad x \geq 1,$$

which means that $\lim_{\alpha \rightarrow \infty} \lambda(\alpha) = 0$. We observe that

$$\lambda'(\alpha) = [(l+2)\eta'(\alpha) - \eta(\alpha)][\eta(\alpha)]^{l+1} e^{-\alpha}, \quad \alpha > 0.$$

Since $\eta'(\alpha) = -1/w'(\eta(\alpha))$, we have

$$\lambda'(\alpha) = -[\eta(\alpha)w'(\eta(\alpha)) + l + 2] \frac{[\eta(\alpha)]^{l+1}}{e^\alpha w'(\eta(\alpha))}.$$

Since

$$(2.1) \quad (xw'(x))' = w'(x) + xw''(x) = w'(x) - x^{l+1}e^{w(x)} < 0, \quad x > 0,$$

there exists $\alpha_* > 0$ such that

$$(2.2) \quad \eta(\alpha)w'(\eta(\alpha)) + l + 2 > 0, \quad 0 < \alpha < \alpha_*,$$

$$(2.3) \quad \eta(\alpha_*)w'(\eta(\alpha_*)) + l + 2 = 0,$$

$$(2.4) \quad \eta(\alpha)w'(\eta(\alpha)) + l + 2 < 0, \quad \alpha > \alpha_*.$$

Consequently, we see that $\lambda'(\alpha) > 0$ for $0 < \alpha < \alpha_*$, $\lambda(\alpha_*) = 0$ and $\lambda'(\alpha) < 0$ for $\alpha > 0$. \square

Recalling (2.1) and the definition of $\psi(x; \alpha)$, we conclude that $\psi(x; \alpha)$ is strictly decreasing in $x \in (0, 1]$ for each fixed $\alpha > 0$. Since $\psi(-x; \alpha) = \psi(x; \alpha)$, we find that

$$\min_{x \in [-1, 1]} \psi(x; \alpha) = \psi(1; \alpha) = \eta(\alpha)w'(\eta(\alpha)) + l + 2.$$

Then, by (2.2)–(2.4), we have the following result immediately.

Lemma 2.1. *The function $\psi(x; \alpha)$ satisfies the following (i)–(iii):*

- (i) *if $0 < \alpha < \alpha_*$, then $\psi(x; \alpha) > 0$ for $x \in [-1, 1]$;*
- (ii) *$\psi(x; \alpha_*) > 0$ for $x \in (-1, 1)$ and $\psi(-1; \alpha_*) = \psi(1; \alpha_*) = 0$;*
- (iii) *if $\alpha > \alpha_*$, then $\psi(x; \alpha)$ has exactly two zeros in $(-1, 1)$, $\psi(-1; \alpha) < 0$ and $\psi(1; \alpha) < 0$.*

Lemma 2.2. *The first eigenvalue $\mu_1(\alpha)$ of (1.8) satisfies the following (i)–(iii):*

- (i) $\mu_1(\alpha) > 0$ for $0 < \alpha < \alpha_*$;
- (ii) $\mu_1(\alpha_*) = 0$;
- (iii) $\mu_1(\alpha) < 0$ for $\alpha > \alpha_*$.

Proof. Let ϕ_1 be an eigenfunction corresponding to $\mu_1(\alpha)$. We recall that $\phi_1(x) \neq 0$ on $(-1, 1)$ and $\phi_1(-1) = \phi_1(1) = 0$.

(i) Assume that $\mu_1(\gamma_1) \leq 0$ for some $\gamma_1 \in (0, \alpha_*)$. Sturm comparison theorem implies that every solution of (1.9) at $\alpha = \gamma_1$ has at least one zero in $[-1, 1]$. This contradicts (i) of Lemma 2.1. Hence, $\mu_1(\alpha) > 0$ for $0 < \alpha < \alpha_*$.

(ii) From (ii) of Lemma 2.1 it follows that $\psi(x; \alpha_*)$ is an eigenfunction corresponding to $\mu_1(\alpha_*)$ and $\mu_1(\alpha_*) = 0$.

(iii) We assume that $\mu_1(\gamma_2) \geq 0$ for some $\gamma_2 > \alpha_*$. Recalling (iii) of Lemma 2.1 and using Sturm comparison theorem, we conclude that every solution of

$$\phi'' + [\lambda(\gamma_2)|x|^l e^{U(x; \gamma_2)} + \mu_1(\gamma_2)]\phi = 0$$

has at least one zero in $(-1, 1)$. On the other hand, the eigenfunction ϕ_1 of (1.8) corresponding to $\mu_1(\beta)$ has no zero in $(-1, 1)$, which is a contradiction. Consequently, $\mu_1(\alpha) < 0$ for $\alpha > \alpha_*$. \square

Lemma 2.3. *The third eigenvalue $\mu_3(\alpha)$ of (1.8) is positive for $\alpha > 0$.*

Proof. Assume that $\mu_3(\alpha) \leq 0$ for some $\alpha > 0$. Let ϕ_3 be an eigenfunction of (1.8) corresponding to $\mu_3(\alpha)$. Then $\phi_3(-1) = \phi_3(1)$ and ϕ_3 has exactly two zeros in $(-1, 1)$. Sturm comparison theorem shows that every solution of (1.9) has at least three zeros in $[-1, 1]$. Lemmas 1.2 and 2.1 imply that $\psi(x; \alpha)$ is a solution of (1.9) and has at most two zeros in $[-1, 1]$. This is a contradiction. Therefore, $\mu_3(\alpha) > 0$ for $\alpha > 0$. \square

3. THE SECOND EIGENVALUE

The purpose of this section is to give a sufficient condition for the second eigenvalue of the linearized problem to the following problem

$$(3.1) \quad \begin{cases} u'' + \lambda|x|^l f(u) = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0 \end{cases}$$

to be negative, where $\lambda > 0$, $l > 0$, $f \in C^1[0, \infty)$, $f(s) > 0$ and $f'(s) \geq 0$ for $s > 0$. Namely we will show the following lemma.

Lemma 3.1. *Assume that, for each sufficiently large $\alpha > 0$, there exist $\lambda(\alpha) > 0$ and $U(x; \alpha)$ such that $U(x; \alpha)$ is a positive even solution of (3.1) at $\lambda = \lambda(\alpha)$. Assume moreover that*

$$(3.2) \quad \liminf_{s \rightarrow \infty} \frac{l(g(s) - 1) - 4}{g(s) + l + 3} > 0,$$

where $g(s) = sf'(s)/f(s)$. Let $\mu_2(\alpha)$ be the second eigenvalue of

$$(3.3) \quad \begin{cases} \phi'' + \lambda(\alpha)|x|^l f'(U(x; \alpha))\phi + \mu\phi = 0, & x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

Then $\mu_2(\alpha) < 0$ for all sufficiently large $\alpha > 0$.

To this end we need the following two lemmas.

Lemma 3.2. *Let ϕ_2 be an eigenfunction corresponding to the second eigenvalue $\mu_2(\alpha)$ of (3.3). Then ϕ_2 is odd, $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(x) \neq 0$ for $x \in (0, 1)$.*

Proof. Let M_1 be the first eigenvalue of

$$\begin{cases} \Phi'' + \lambda(\alpha)|x|^l f'(U(x; \alpha))\Phi + M\Phi = 0, & x \in (0, 1), \\ \Phi(0) = \Phi(1) = 0 \end{cases}$$

and let Φ_1 be an eigenfunction corresponding to M_1 . Then $\Phi_1(0) = \Phi_1(1) = 0$ and $\Phi_1(x) \neq 0$ on $(0, 1)$. Set

$$\Phi(x) = \begin{cases} \Phi_1(x), & x \in [0, 1], \\ -\Phi_1(-x), & x \in [-1, 0). \end{cases}$$

Noting that

$$\lim_{x \rightarrow -0} \Phi''(x) = \lim_{x \rightarrow -0} (-\Phi_1''(-x)) = -\Phi_1''(0) = 0,$$

we easily check that Φ is a solution of

$$\begin{cases} \Phi'' + \lambda(\alpha)|x|^l f'(U(x; \alpha))\Phi + M_1\Phi = 0, & x \in (-1, 1), \\ \Phi(-1) = \Phi(1) = 0, \end{cases}$$

and Φ is odd, $\Phi(x) \neq 0$ on $(0, 1)$ and $\Phi(0) = 0$. Therefore, M_1 is an eigenvalue of (3.3) and Φ is an eigenfunction corresponding to M_1 . Since Φ has exactly one zero in $(-1, 1)$, M_1 must be μ_2 and hence $\phi_2(x)$ must be $c\Phi(x)$ for some $c \neq 0$. \square

Lemma 3.3. *Assume that $w \in C[a, b]$ is positive and concave on (a, b) . Let $\rho \in (0, 1/2)$. Then $w(x) \geq \rho \max_{\xi \in [a, b]} w(\xi)$ for $x \in [(1 - \rho)a + \rho b, \rho a + (1 - \rho)b]$.*

Proof. We take $c \in [a, b]$ for which $w(c) = \max_{\xi \in [a, b]} w(\xi)$. Then $w(c) > 0$. Since w is positive and concave on (a, b) , we have

$$w(x) \geq \frac{w(c)(x - a)}{c - a} \geq \frac{w(c)(x - a)}{b - a} =: l_1(x), \quad x \in [a, c],$$

and

$$w(x) \geq \frac{w(c)(b - x)}{b - c} \geq \frac{w(c)(b - x)}{b - a} =: l_2(x), \quad x \in [c, b].$$

Hence $w(x) \geq \min\{l_1(x), l_2(x)\}$ on $[a, b]$. We conclude that if $x \in [(1 - \rho)a + \rho b, (a + b)/2]$, then

$$\min\{l_1(x), l_2(x)\} = l_1(x) \geq l_1((1 - \rho)a + \rho b) = \rho w(c),$$

and if $x \in [(a + b)/2, \rho a + (1 - \rho)b]$, then

$$\min\{l_1(x), l_2(x)\} = l_2(x) \geq l_2(\rho a + (1 - \rho)b) = \rho w(c).$$

The proof is complete. \square

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let $\alpha > 0$ be sufficiently large. We use the following comparison function $y(x)$ introduced in [19]:

$$y(x) = xU(x; \alpha) - (x - 1)^2U'(x; \alpha).$$

This function $y(x)$ satisfies $y(0) = y(1) = 0$, $y(x) > 0$ on $(0, 1)$, and

$$y'' + \lambda(\alpha)|x|^l f'(U(x; \alpha))y = \lambda(\alpha)x^{l-1}H(x; \alpha)f(U(x; \alpha)), \quad x \in (0, 1],$$

where

$$H(x; \alpha) = [g(U(x; \alpha)) + l + 3]x^2 - 2(l + 2)x + l.$$

Let ϕ_2 be an eigenfunction corresponding to $\mu_2(\alpha)$. From Lemma 3.2 it follows that $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(x) \neq 0$ for $x \in (0, 1)$. Without loss of generality, we may assume that $\phi_2(x) > 0$ for $x \in (0, 1)$ and $\max_{\xi \in [0, 1]} \phi_2(\xi) = 1$. We observe that

$$(y'\phi_2 - y\phi_2')' = \mu_2(\alpha)\phi_2 y + \lambda(\alpha)x^{l-1}H(x; \alpha)f(U(x; \alpha))\phi_2, \quad x \in (0, 1].$$

Integrating this equality on $(0, 1)$, we obtain

$$(3.4) \quad \mu_2(\alpha) \int_0^1 \phi_2(x)y(x)dx + \lambda(\alpha) \int_0^1 x^{l-1}H(x; \alpha)f(U(x; \alpha))\phi_2(x)dx = 0.$$

Since

$$\begin{aligned} H(x; \alpha) &= [g(U(x; \alpha)) + l + 3] \left(x - \frac{l + 2}{g(U(x; \alpha)) + l + 3} \right)^2 \\ &\quad + \frac{l[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l + 3} \\ &\geq \frac{l[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l + 3}, \end{aligned}$$

we have

$$(3.5) \quad \int_0^1 x^{l-1}H(x; \alpha)f(U(x; \alpha))\phi_2(x)dx \geq \int_0^1 x^{l-1} \frac{l[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l + 3} f(U(x; \alpha))\phi_2(x)dx.$$

By (3.2), there exist $\delta > 0$ and sufficiently large $s_0 > 0$ such that

$$\frac{l(g(s) - 1) - 4}{g(s) + l + 3} \geq \delta, \quad s \geq s_0.$$

Since $U''(x; \alpha) = -\lambda(\alpha)|x|^l f(U(x; \alpha)) < 0$ on $(0, 1]$, we find that $U'(x; \alpha)$ is decreasing in $x \in (0, 1]$. From $U'(0; \alpha) = 0$, it follows that $U'(x; \alpha) < 0$ for $x \in (0, 1]$, which implies that $U(x; \alpha)$ is also decreasing in $x \in (0, 1]$. Now let $\alpha > s_0$. Then there exists $x(\alpha) \in (0, 1)$ such that

$U(x; \alpha) \geq s_0$ for $x \in [0, x(\alpha)]$ and $U(x; \alpha) < s_0$ for $x \in (x(\alpha), 1]$. Since $U(x; \alpha)$ is concave on $(0, 1)$, we conclude that

$$U(x; \alpha) \geq \alpha(1 - x), \quad x \in [0, 1],$$

which shows that if $x \in [0, (\alpha - s_0)/\alpha]$, then $U(x; \alpha) \geq s_0$. Therefore, $x(\alpha) \geq (\alpha - s_0)/\alpha$, which implies

$$(3.6) \quad \lim_{\alpha \rightarrow \infty} x(\alpha) = 1.$$

We take $s_1 \geq s_0$ for which $x(\alpha) \geq 3/4$ for $\alpha \geq s_1$. If $\alpha \geq s_1$, then

$$(3.7) \quad \begin{aligned} & \int_0^{x(\alpha)} x^{l-1} \frac{l[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l + 3} f(U(x; \alpha)) \phi_2(x) dx \\ & \geq \int_0^{x(\alpha)} x^{l-1} \delta f(s_0) \phi_2(x) dx \geq \delta f(s_0) \int_{1/4}^{3/4} x^{l-1} \phi_2(x) dx. \end{aligned}$$

Recalling $\max_{\xi \in [0, 1]} \phi_2(\xi) = 1$, we have

$$(3.8) \quad \begin{aligned} & \int_{x(\alpha)}^1 x^{l-1} \frac{l[g(U(x; \alpha)) - 1] - 4}{g(U(x; \alpha)) + l + 3} f(U(x; \alpha)) \phi_2(x) dx \\ & \geq -(l + 4) \int_{x(\alpha)}^1 x^{l-1} \frac{f(U(x; \alpha)) \phi_2(x)}{g(U(x; \alpha)) + l + 3} dx \\ & \geq -(l + 4) \int_{x(\alpha)}^1 \frac{f(s_0)}{l + 3} dx \\ & = -\frac{(l + 4)f(s_0)}{l + 3} (1 - x(\alpha)), \quad \alpha \geq s_0. \end{aligned}$$

Now we will show that there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. Assume to the contrary that there exists $\{\alpha_n\}_{n=1}^\infty$ such that $\mu_2(\alpha_n) \geq 0$ and $\alpha_n \geq s_1$ for $n \in \mathbf{N}$ and $\lim_{n \rightarrow \infty} \alpha_n = \infty$. Since $\phi_2(x) > 0$ and $\phi_2''(x) = -|x|^l f'(U(x; \alpha_n)) \phi_2 - \mu_2(\alpha_n) \phi_2 \leq 0$ on $(0, 1)$, we find that ϕ_2 is concave on $(0, 1)$ when $\alpha = \alpha_n$. From Lemma 3.3 with $\rho = 1/4$, $a = 0$ and $b = 1$, it follows that

$$(3.9) \quad \phi_2(x) \geq \frac{1}{4} \max_{\xi \in [0, 1]} \phi_2(\xi) = \frac{1}{4} \quad \text{for } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \alpha = \alpha_n.$$

Combining (3.4) with (3.5), (3.7)–(3.9), we conclude that

$$\begin{aligned} 0 & \geq -\mu_2(\alpha_n) \int_0^1 \phi_2(x) y(x) dx \\ & \geq \lambda(\alpha_n) f(s_0) \left[\frac{\delta}{4} \int_{1/4}^{3/4} x^{l-1} dx - \frac{l + 4}{l + 3} (1 - x(\alpha_n)) \right], \end{aligned}$$

which implies

$$\frac{l+4}{l+3}(1-x(\alpha_n)) \geq \frac{\delta}{4} \int_{1/4}^{3/4} x^{l-1} dx > 0, \quad n \in \mathbf{N}.$$

This contradicts the fact (3.6). Consequently, there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. This completes the proof of Lemma 3.1. \square

4. EXISTENCE OF ANOTHER LARGE SOLUTION

In this section we give a criterion for the existence of a large positive solution if there exists a positive even solution with the Morse index 2.

We consider the following problem

$$(4.1) \quad \begin{cases} u'' + h(x)f(u) = 0, & x \in (-1, 1), \\ u(-1) = u(1) = 0. \end{cases}$$

Throughout this section, the following conditions are assumed to hold: $h \in C[-1, 1]$, $h(x) \geq 0$ for $x \in [-1, 1]$, $h(x)$ has at most finite zeros in $[-1, 1]$, $f \in C^1[0, \infty)$, $f(s) > 0$, $f'(s) \geq 0$ for $s \geq 0$, and

$$(4.2) \quad \lim_{s \rightarrow \infty} \frac{f(s)}{s} = \infty.$$

The purpose of this section is to prove the following existence result which will be used in the proof of Theorem 1.1.

Lemma 4.1. *Assume that (4.1) has a positive solution U for which the Morse index of U is 2 and U is nondegenerate. Then (4.1) has a positive solution u such that $u \not\equiv U$ and $Mf(\|u\|_\infty) > \|U\|_\infty$, where*

$$M = \int_{-1}^1 \int_{-1}^x h(t) dt dx.$$

Here, the Morse index of U is the number of negative eigenvalues μ of the problem

$$(4.3) \quad \begin{cases} \phi'' + h(x)f'(U(x))\phi + \mu\phi = 0, & x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

To prove Lemma 4.1, we extend the domain of $f(s)$ satisfying $f \in C^1(\mathbf{R})$ and $f(x) > 0$ for $x \in \mathbf{R}$. We also extend the domain of $h(x)$ satisfying

$$h \in C[-1, 1], \quad h(x) \geq 0 \text{ for } x \geq -1 \text{ and } \liminf_{x \rightarrow \infty} h(x) > 0.$$

We denote by $u(x; \beta)$ the solution of the initial value problem

$$\begin{cases} u'' + h(x)f(u) = 0, \\ u(-1) = 0, \quad u'(-1) = \beta, \end{cases}$$

where $\beta > 0$ is a parameter. From a general theory on ordinary differential equations (see, for example, [6]), it follows that the solution $u(x; \beta)$ exists on $[-1, \infty)$, it is unique, and $u(x; \beta)$, $u'(x; \beta)$ are C^1 functions on the set $[-1, \infty) \times (0, \infty)$. By the same argument as in the proof of Lemma 2.1 in [19], we easily see that, for each $\beta > 0$, $u(x; \beta)$ has a zero in $[-1, \infty)$. For each $\beta > 0$, we denote the first zero of $u(x; \beta)$ in $(-1, \infty)$ by $z(\beta)$. Since $u(x; \beta) > 0$ for $x \in (-1, z(\beta))$, by the uniqueness of the initial value problem, we have $u'(z(\beta); \beta) < 0$. Therefore we conclude that

$$u(z(\beta); \beta) = 0, \quad u'(z(\beta); \beta) < 0.$$

The implicit function theorem shows that $z \in C^1(0, \infty)$ and

$$(4.4) \quad z'(\beta) = -\frac{\frac{\partial u}{\partial \beta}(z(\beta); \beta)}{u'(z(\beta); \beta)}.$$

By a general theory on ordinary differential equations (see, for example, [6]), we note that $\frac{\partial u}{\partial \beta}(x; \beta)$ is a unique solution of the initial value problem

$$(4.5) \quad \begin{cases} v'' + h(x)f'(u)v = 0, \\ v(-1) = 0, \quad v'(-1) = 1, \end{cases}$$

where $u = u(x; \beta)$.

Lemma 4.2. *There exists $\beta^* > 0$ such that $z(\beta) < 1$ for $\beta > \beta^*$.*

Proof. Assume that there exists $\{\beta_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} \beta_n = \infty$, $\beta_n > 0$ and $z(\beta_n) \geq 1$ for $n \in \mathbf{N}$. Set $u_n = u(x; \beta_n)$. Then $u_n(x) > 0$ for $x \in (-1, 1)$. Integrating $u_n'' + h(x)f(u_n) = 0$ on $[-1, x]$ and integrating it on $[-1, 1]$ again, we have

$$(4.6) \quad 2\beta_n = \int_{-1}^1 \int_{-1}^x h(s)f(u_n(s))dsdx \leq f(\|u_n\|_\infty) \int_{-1}^1 \int_{-1}^x h(s)dsdx.$$

Letting $n \rightarrow \infty$ in (4.6), we obtain

$$(4.7) \quad \lim_{n \rightarrow \infty} \|u_n\|_\infty = \infty.$$

From Lemma 3.3 with $a = -1$, $b = 1$ and $\rho = 1/4$, it follows that

$$(4.8) \quad u_n(x) \geq \frac{1}{4}\|u_n\|_\infty > 0, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Let ν_1 be the first eigenvalue of

$$\begin{cases} \phi'' + \nu h(x)\phi = 0, & x \in (-1/2, 1/2), \\ \phi(-1/2) = \phi(1/2) = 0. \end{cases}$$

Then $\nu_1 > 0$. By (4.2), there exists $s_1 > 0$ such that

$$\frac{f(s)}{s} > \nu_1, \quad s > s_1.$$

By (4.7), there exists $n_1 > 0$ such that $\|u_n\|_\infty > 4s_1$ for $n \geq n_1$. From (4.8) it follows that if $n \geq n_1$, then

$$h(x) \frac{f(u_n(x))}{u_n(x)} > \nu_1 h(x), \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Since u_n is a solution of

$$u_n'' + h(x) \frac{f(u_n(x))}{u_n(x)} u_n = 0,$$

Sturm comparison theorem implies that u_n has at least one zero in $(-1/2, 1/2)$. This contradicts (4.8). Therefore, there exists $\beta^* > 0$ such that $z(\beta) < 1$ for $\beta > \beta^*$. \square

Lemma 4.3. *Assume that (4.1) has a positive solution U for which the Morse index of U is 2 and U is nondegenerate. Then $z'(U'(-1)) > 0$.*

Proof. First we note that $U(x) = u(x; U'(-1))$ for $x \in [-1, 1]$ and $z(U'(-1)) = 1$. Let μ_2 and μ_3 be the second and third eigenvalues of (4.3), respectively. Then $\mu_2 < 0 < \mu_3$. Let ϕ_2 and ϕ_3 be eigenfunctions corresponding to μ_2 and μ_3 , respectively. Let v be the solution of (4.5) with $u = U$. We recall that $v(x) \equiv \frac{\partial u}{\partial \beta}(x; U'(-1))$. Since ϕ_2 has exactly one zero in $(-1, 1)$, Sturm comparison theorem implies that v has at least two zeros in $(-1, 1)$. If v has three zeros in $(-1, 1]$, then, by Sturm comparison theorem again, ϕ_3 has at least three zeros in $(-1, 1)$, which is a contradiction. Therefore, v has exactly two zeros in $(-1, 1)$ and $v(1) \neq 0$. Since $v'(-1) = 1 > 0$, we conclude that $v(1) > 0$. Since $\frac{\partial u}{\partial \beta}(z(U'(-1)); U'(-1)) = v(1) > 0$ and $u'(z(U'(-1)); U'(-1)) = U'(1) < 0$, by (4.4), we obtain $z'(U'(-1)) > 0$. \square

Now we are ready to show Lemma 4.1.

Proof of Lemma 4.1. By Lemma 4.2, there exists $\beta^* > 0$ such that $z(\beta) < 1$ for $\beta > \beta^*$. Hence, by Lemma 4.3 and $z(U'(-1)) = 1$, there exists $\beta_0 \in (U'(-1), \beta^*)$ such that $z(\beta_0) = 1$. Then $u := u(x; \beta_0)$ is a positive solution of (4.1). Since $u'(-1) = \beta_0 > U'(-1)$, we conclude that $u \neq U$, by the uniqueness of the initial value problem. Integrating

$u'' + h(x)f(u) = 0$ on $[-1, x]$ and integrating it on $[-1, 1]$ again, we have

$$2\beta_0 = \int_{-1}^1 \int_{-1}^x h(t)f(u(t))dt dx \leq Mf(\|u\|_\infty).$$

Let $c \in (-1, 1)$ satisfy $U(c) = \|U\|_\infty$. Since U is concave on $(-1, 1)$, we have

$$U(x) \geq \frac{\|U\|_\infty}{c+1}(x+1), \quad x \in [-1, c].$$

Hence,

$$U'(-1) = \lim_{x \rightarrow -1} \frac{U(x) - U(-1)}{x+1} = \lim_{x \rightarrow -1} \frac{U(x)}{x+1} \geq \frac{\|U\|_\infty}{c+1} \geq \frac{\|U\|_\infty}{2}.$$

Consequently,

$$Mf(\|u\|_\infty) \geq 2\beta_0 > 2U'(-1) \geq \|U\|_\infty.$$

□

5. PROOF OF THE MAIN RESULT

In this section we give a proof of Theorem 1.1.

Lemma 2.2 means (i) and (ii) of Theorem 1.1. Moreover, since $\mu_2(\alpha) > \mu_1(\alpha)$, we have

$$(5.1) \quad \mu_2(\alpha) > 0, \quad 0 < \alpha \leq \alpha_*.$$

When $f(s) = e^s$, we have $g(s) := sf'(s)/f(s) = s$ and

$$\liminf_{s \rightarrow \infty} \frac{l(g(s) - 1) - 4}{g(s) + l + 3} = l > 0.$$

From Lemma 3.1 it follows that $\mu_2(\alpha) < 0$ for all sufficiently large $\alpha > 0$. Hence, by (5.1), there exist α_1 and α_3 such that $\alpha_* < \alpha_1 \leq \alpha_3$ such that

$$(5.2) \quad \mu_2(\alpha_1) = 0, \quad \mu_2(\alpha) > 0, \quad 0 < \alpha < \alpha_1$$

and

$$\mu_2(\alpha_3) = 0, \quad \mu_2(\alpha) < 0, \quad \alpha > \alpha_3.$$

Therefore, Lemma 2.3 implies (vi) and (vii) of Theorem 1.1. From Lemma 2.2 and (5.2), it follows that (iii) and (iv) of Theorem 1.1 hold.

Now we will show (v). To this end, we define $T(\alpha, v)$ by

$$T(\alpha, v) = \int_{-1}^1 G(x, y) \lambda(\alpha) |y|^l e^{U(y; \alpha)} (e^{v(y)} - 1) dy,$$

where $G(x, y)$ is a Green's function of the operator $L[v] = -v''$ with $v(-1) = v(1) = 0$:

$$G(x, y) = \begin{cases} (1+x)(1-y)/2, & -1 \leq x \leq y \leq 1, \\ (1-x)(1+y)/2, & -1 \leq y \leq x \leq 1. \end{cases}$$

Then (1.1) can be rewritten as

$$(5.3) \quad v - T(\alpha, v) = 0.$$

We note that (5.3) has a solution $v = 0$ and if v is a solution of (5.3), then $u(x) = U(x; \alpha) + v(x)$ is a solution of (1.1) at $\lambda = \lambda(\alpha)$.

Lemma 5.1. *Let $\gamma(\alpha)$ be the sum of algebraic multiplicities of all the eigenvalues of $T'_v(\alpha, 0)$ contained in $(1, \infty)$. Then $m(\alpha) = \gamma(\alpha)$.*

Proof. First we note that an eigenvalue ν of $T'_v(\alpha, 0)$ with $\nu > 1$ is an eigenvalue of the problem

$$(5.4) \quad \begin{cases} \psi'' + \frac{1}{\nu} \lambda(\alpha) |x|^l e^{U(x; \alpha)} \psi = 0, & x \in (-1, 1), \\ \psi(-1) = \psi(1) = 0. \end{cases}$$

We conclude that (5.4) has eigenvalues $\{\nu_k(\alpha)\}_{k=1}^\infty$ for which

$$\nu_1(\alpha) > \nu_2(\alpha) > \cdots > \nu_k(\alpha) > \nu_{k+1}(\alpha) > \cdots > 0, \quad \lim_{k \rightarrow \infty} \nu_k(\alpha) = 0,$$

no other eigenvalues, an eigenfunction ψ_k corresponding to $\nu_k(\alpha)$ is unique up to a constant, and ψ_k has exactly $k - 1$ zeros in $(-1, 1)$.

Next we will show that $\nu_k(\alpha) > 1$ implies $\mu_k(\alpha) < 0$. Assume that $\mu_k(\alpha) > 1$ and $\mu_k(\alpha) \geq 0$. Then

$$\frac{1}{\nu_k(\alpha)} \lambda(\alpha) |x|^l e^{U(x; \alpha)} < \lambda(\alpha) |x|^l e^{U(x; \alpha)} + \mu_k(\alpha).$$

Sturm comparison theorem implies that an eigenfunction ϕ_k corresponding to $\mu_k(\alpha)$ has at least k zeros in $(-1, 1)$. This is a contradiction. Hence, $\nu_k(\alpha) > 1$ implies $\mu_k(\alpha) < 0$.

Finally we will prove that $\mu_k(\alpha) < 0$ implies $\nu_k(\alpha) > 1$. Assume that $\mu_k(\alpha) < 0$ and $\nu_k(\alpha) \leq 1$. Since

$$\frac{1}{\nu_k(\alpha)} \lambda(\alpha) |x|^l e^{U(x; \alpha)} > \lambda(\alpha) |x|^l e^{U(x; \alpha)} + \mu_k(\alpha),$$

By Sturm comparison theorem again, we conclude that an eigenfunction ψ_k corresponding to $\nu_k(\alpha)$ has at least one k zeros in $(-1, 1)$, which is a contradiction. Then $\mu_k(\alpha) < 0$ implies $\nu_k(\alpha) > 1$.

Consequently, $m(\alpha) = \gamma(\alpha)$. □

Lemma 5.2. *For each sufficiently small $\varepsilon > 0$, there exists $(\alpha_\varepsilon, v_\varepsilon)$ such that $\alpha_1 - \varepsilon \leq \alpha_\varepsilon \leq \alpha_3 + \varepsilon$, $v_\varepsilon \in C[-1, 1]$, and*

$$v_\varepsilon - T(\alpha_\varepsilon, v_\varepsilon) = 0, \quad \|v_\varepsilon\|_\infty \leq \varepsilon, \quad v_\varepsilon \neq 0.$$

Proof. Assume there exists $\varepsilon > 0$ such that

$$v - T(\alpha, v) \neq 0 \quad \text{for } \alpha_1 - \varepsilon \leq \alpha \leq \alpha_3 + \varepsilon, \quad v \in B_\varepsilon(0) - \{0\},$$

where $B_\varepsilon(0) = \{v \in C[-1, 1] : \|v\|_\infty < \varepsilon\}$. Since $T(\alpha, v)$ is a compact operator on $C[0, 1]$ for each fixed $\alpha > 0$, Leray-Schauder degree $\deg_{\text{LS}}(I - T(\alpha, \cdot), B_\varepsilon(0), 0)$ is well defined in $C[-1, 1]$. By the homotopy invariance of the Leray Schauder degree, we conclude that

$$\deg_{\text{LS}}(I - T(\alpha, \cdot), B_\varepsilon(0), 0) \text{ is constant for } \alpha_1 - \varepsilon \leq \alpha \leq \alpha_3 + \varepsilon.$$

It is known (for example, [1, Theorem 3.20]) that

$$\deg_{\text{LS}}(I - T(\alpha_1 - \varepsilon, \cdot), B_\varepsilon(0), 0) = (-1)^{\gamma(\alpha_1 - \varepsilon)}$$

and

$$\deg_{\text{LS}}(I - T(\alpha_3 + \varepsilon, \cdot), B_\varepsilon(0), 0) = (-1)^{\gamma(\alpha_3 + \varepsilon)},$$

where $\gamma(\alpha)$ is as in Lemma 5.1. Lemma 5.1 implies that

$$\gamma(\alpha_1 - \varepsilon) = m(\alpha_1 - \varepsilon) = 1$$

and

$$\gamma(\alpha_3 + \varepsilon) = m(\alpha_3 + \varepsilon) = 2,$$

which means that

$$\deg_{\text{LS}}(I - T(\alpha_1 - \varepsilon, \cdot), B_\varepsilon(0), 0) = -1$$

and

$$\deg_{\text{LS}}(I - T(\alpha_3 + \varepsilon, \cdot), B_\varepsilon(0), 0) = 1.$$

This contradicts the homotopy invariance of the Leray-Schauder degree. \square

Now we are ready to prove (v) of Theorem 1.1. Let $\{(\alpha_\varepsilon, v_\varepsilon)\}$ be as in Lemma 5.2. Since $\alpha_\varepsilon \in [\alpha_1 - \varepsilon, \alpha_3 + \varepsilon]$, there exists a subsequence of $\{(\alpha_\varepsilon, v_\varepsilon)\}$, again denoted by $\{(\alpha_\varepsilon, v_\varepsilon)\}$ such that

$$\alpha_\varepsilon \rightarrow \alpha_2, \quad v_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow +0$$

for some $\alpha_2 \in [\alpha_1, \alpha_3]$. Consequently, $(\lambda(\alpha_2), U(x; \alpha_2))$ is a bifurcation point. Clearly, $U(x; \alpha_2)$ is degenerate. Moreover, $u_\varepsilon(x) := U(x; \alpha_\varepsilon) + v_\varepsilon(x)$ is a solution of (1.1). By recalling that $\lambda'(\alpha) < 0$ for $\alpha > \alpha_*$, there is no even solution u of (1.1) at $\lambda = \lambda(\alpha)$ such that $\|u\|_\infty > \alpha_*$ except $U(x; \alpha)$. Since

$$\alpha_\varepsilon - \varepsilon \leq \|u_\varepsilon\|_\infty \leq \alpha_\varepsilon + \varepsilon,$$

we conclude that u_ε is a non-even solution of (1.1), and hence (v) of Theorem 1.1 holds.

Finally, we give a proof of the remaining part of Theorem 1.1, that is, we will show that, for each $\lambda \in (0, \lambda(\alpha_3))$, problem (1.1) has a positive non-even solution $u(x)$ which satisfies $\lim_{\lambda \rightarrow +0} \|u\|_\infty = \infty$. Let $\lambda \in (0, \lambda(\alpha_3))$. Then, by Proposition 1.1, there exists $\alpha_\lambda > \alpha_3$ such that $\lambda(\alpha_\lambda) = \lambda$, $\lim_{\lambda \rightarrow +0} \alpha_\lambda = \infty$ and $\lim_{\lambda \rightarrow +0} \lambda(\alpha_\lambda) = 0$. From (vii) of Theorem 1.1 it follows that $m(\alpha_\lambda) = 2$ and $U(x; \alpha_\lambda)$ is nondegenerate. Lemma 4.1 implies that (1.1) has a positive solution u such that $u(x) \neq U(x; \alpha_\lambda)$ and

$$\lambda(\alpha_\lambda) M e^{\|u\|_\infty} > \alpha_\lambda$$

for some constant $M > 0$, which shows $\lim_{\lambda \rightarrow +0} \|u\|_\infty = \infty$. Recalling that (1.1) has at most two positive even solutions, we conclude that u is a positive non-even solution. This completes the proof of Theorem 1.1.

6. PROOF OF THE SECOND MAIN RESULT

In this section we prove Propositions 1.2, 1.3 and Theorem 1.2.

Let w be a unique solution of the initial value problem

$$\begin{cases} w'' + |x|^l(w+1)^p = 0, & x > 0, \\ w(0) = w'(0) = 0. \end{cases}$$

Since w is concave when $w(x) > -1$, there exist $x_1 > 0$ such that $-1 < w(x) < 0$, $w'(x) < 0$, $w''(x) < 0$ for $x \in (0, x_1)$, $w(x_1) = -1$, and $w'(x_1) < 0$. Hence, there exists the inverse function η of $-w(x)$. It follows that $\eta \in C^2(0, 1]$, $\eta(t) > 0$, $\eta'(t) > 0$ for $t \in (0, 1]$, $\eta(0) = 0$, and $\eta(1) = x_1$. We set

$$(6.1) \quad \lambda(\alpha) = (\alpha + 1)^{1-p} \left[\eta \left(\frac{\alpha}{\alpha + 1} \right) \right]^{l+2}$$

and

$$(6.2) \quad U(x; \alpha) = (\alpha + 1) w \left(\eta \left(\frac{\alpha}{\alpha + 1} \right) |x| \right) + \alpha.$$

Then $(\lambda(\alpha), U(x; \alpha))$ is a Korman solution of (1.10), that is, for each $\alpha > 0$, $U(x; \alpha)$ satisfies $\|U\|_\infty = \alpha$ and is a positive even solution of (1.10) at $\lambda = \lambda(\alpha)$. The form of $U(x; \alpha)$ is not exactly same as in the paper by Korman [14], but they are essentially same.

Proof of Proposition 1.3. By the definition, it is easy to check that $\lambda(\alpha) \in C^2(0, \infty)$, $U(x; \alpha) \in C^2([-1, 1] \times (0, \infty))$ and (1.7) holds, because of $p > 1$. Set

$$\beta = \eta \left(\frac{\alpha}{\alpha + 1} \right).$$

Then $-w(\beta) = \alpha/(\alpha + 1)$, that is, $\alpha = -w(\beta)/(w(\beta) + 1)$. Hence we have

$$\lambda(\alpha) = (w(\beta) + 1)^{p-1} \beta^{l+2}.$$

We note that

$$\frac{d\beta}{d\alpha} = \frac{1}{(\alpha + 1)^2} \eta' \left(\frac{\alpha}{\alpha + 1} \right) > 0, \quad \alpha > 0.$$

We observe that

$$\lambda'(\alpha) = (w(\beta) + 1)^{p-2} \beta^{l+1} [(p-1)\beta w'(\beta) + (l+2)(w(\beta) + 1)] \frac{d\beta}{d\alpha}.$$

We also note that

$$(6.3) \quad W(x) := (p-1)xw'(x) + (l+2)(w(x) + 1)$$

is strictly decreasing on $(0, x_1)$, since

$$(xw'(x))' = w'(x) + xw''(x) < 0, \quad x \in (0, x_1).$$

Since $W(0) = l+2 > 0$ and $W(x_1) = (p-1)x_1w'(x_1) < 0$, there exists $\beta_* \in (0, x_1)$ such that

$$(6.4) \quad W(x) > 0, \quad 0 < x < \beta_*,$$

$$(6.5) \quad W(\beta_*) = 0,$$

$$(6.6) \quad W(x) < 0, \quad \beta_* < x < x_1.$$

Set $\alpha_* = -w(\beta_*)/(w(\beta_*) + 1)$. Then we conclude that $\lambda'(\alpha) > 0$ for $0 < \alpha < \alpha_*$, $\lambda(\alpha_*) = 0$ and $\lambda'(\alpha) < 0$ for $\alpha > \alpha_*$. \square

To prove Proposition 1.2, we need the following lemma.

Lemma 6.1. *For each $\alpha > 0$, there exists a unique (λ, u) such that $\lambda > 0$ and u is a positive even solution of (1.10) and $\|u\|_\infty = \alpha$. In particular, all positive even solutions of (1.10) can be written as (6.1)–(6.2).*

Proof. Let $\alpha > 0$ be fixed. We consider the initial value problem

$$(6.7) \quad \begin{cases} u'' + \lambda|x|^l(u+1)^p = 0, \\ u(0) = \alpha, \quad u'(0) = 0. \end{cases}$$

We note that

$$u(x; \lambda) := (\alpha + 1)w(\lambda^{\frac{1}{l+2}}(\alpha + 1)^{\frac{p-1}{l+2}}|x|) + \alpha$$

is a solution of (6.7). By the uniqueness of the initial value problem, we conclude that $u(x; \lambda)$ is a unique solution of (6.7). We note that $u(1; \lambda) = 0$ if and only if $\lambda = \lambda(\alpha)$. It follows that $u(x; \lambda)$ is a positive even solution of (1.10) if and only if $\lambda = \lambda(\alpha)$, which means that there exists a unique $\lambda > 0$ such that $u(x; \lambda)$ is a solution of (1.10). When $\lambda = \lambda(\alpha)$, we find that $u(x; \lambda) = U(x; \alpha)$. \square

Proof of Proposition 1.2. Set $\lambda_* = \lambda(\alpha_*)$. Then Proposition 1.2 follows immediately from Proposition 1.3 and Lemma 6.1. \square

Now we set

$$\begin{aligned} \psi(x; \alpha) &:= xU'(x; \alpha) + \frac{l+2}{p-1}[U(x; \alpha) + 1] \\ &= \frac{\alpha+1}{p-1}W\left(\eta\left(\frac{\alpha}{\alpha+1}\right)|x|\right), \end{aligned}$$

where W is the function defined by (6.3). Then it is easy to check that $\psi(x; \alpha)$ is a solution of the linearized equation

$$\psi'' + \lambda(\alpha)|x|^l p(U(x; \alpha) + 1)^{p-1} \psi = 0.$$

Recalling that $W(x)$ is strictly decreasing in $x \in (0, x_1)$, we conclude that $\psi(x; \alpha)$ is also strictly decreasing in $x \in (0, 1)$ for each fixed $\alpha > 0$, and hence

$$\min_{x \in [-1, 1]} \psi(x; \alpha) = \psi(1; \alpha) = \frac{\alpha+1}{p-1}W\left(\eta\left(\frac{\alpha}{\alpha+1}\right)\right).$$

Hereafter, let $\mu_k(\alpha)$ be the k -th eigenvalue of

$$(6.8) \quad \begin{cases} \phi'' + \lambda(\alpha)|x|^l p(U(x; \alpha) + 1)^{p-1} \phi + \mu \phi = 0, & x \in (-1, 1), \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

By (6.4)–(6.6), in the same way as in Section 2, we have the following result.

Lemma 6.2. *The following (i)–(iv) hold:*

- (i) $\mu_1(\alpha) > 0$ for $0 < \alpha < \alpha_*$;
- (ii) $\mu_1(\alpha_*) = 0$;
- (iii) $\mu_1(\alpha) < 0$ for $\alpha > \alpha_*$;
- (iv) $\mu_3(\alpha) > 0$ for $\alpha > 0$.

When $f(s) = (s+1)^p$, we have $g(s) = sf'(s)/f(s) = ps/(s+1)$ and then

$$\lim_{s \rightarrow \infty} \frac{l(g(s) - 1) - 4}{g(s) + l + 3} = \frac{l(p-1) - 4}{p + l + 3}.$$

Therefore, if $(p-1)l > 4$, then Lemma 3.1 shows that $\mu_2(\alpha) < 0$ for all sufficiently large $\alpha > 0$.

In the same way as in Section 5, we can show (i)–(vii) of Theorem 1.2. By using Lemma 4.1 and the same argument as in Section 5, we conclude that if $0 < \lambda < \lambda(\alpha_3)$, then (1.10) has a positive non-even solution u such that $\lim_{\lambda \rightarrow +0} \|u\| = \infty$. This completes the proof of Theorem 1.2.

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